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DECAY ESTIMATES OF VISCOSITY SOLUTIONS OF NONLINEAR PARABOLIC PDES AND APPLICATIONS

SILVANA MARCHI

The aim of this paper is to establish decay estimates for viscosity solutions of nonlinear parabolic PDEs. As an application we prove existence and uniqueness for time almost periodic viscosity solutions.

1. Background and motivation

In this paper we shall deal with viscosity solutions of the Cauchy problem

$$\begin{cases} (E) & u_t + H(t,x,u,Du,D^2u) = f(t,x), \;\; (t,x) \in (0,T) \times \mathbb{R}^N \\ (IC) & u(0,x) = u_o(x), \;\; x \in \mathbb{R}^N \end{cases} \tag{1}$$

in which T is a given positive number, H is a real-valued function defined on $[0,T) \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times S(N)$, where $S(N)$ denotes the set of real $N \times N$ matrices, u_o and f are given real-valued continuous functions defined on $\mathbb{R}^N$, and on $[0,T) \times \mathbb{R}^N$ respectively; u is the real-valued unknown function and u_t, Du and D^2u denote the partial derivative with respect to t, the gradient with respect to x and the Hessian matrix with respect to x respectively.

We recall that the notion of viscosity solution was introduced by Crandall and Lions [9] in the case of first-order PDEs. This generalized solution need not be differentiable anywhere, as the only regularity required in the definition

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is continuity. The authors (see also [7]) studied existence and uniqueness of viscosity solutions, the uniqueness following from comparison theorems. The method was improved by several authors (see among others Souganidis [18] and Barles [2]).

The generalization of the comparison results to the case of second order PDEs was first done by Lions [16] and Jensen [15] and then improved by several authors, in particular by Crandall, Ishii, Lions, see [8]. See also Ishii and Kobayasi [11] using a Osgood type condition and Bieske [3] in the case of vector fields. A uniqueness statement obviously follows from the comparison principle.

Our purpose in this paper is to exploit the comparison result as stated in [8] on a bounded domain to prove it for an unbounded domain and, consequently, to prove decay estimates for viscosity sub/super solutions of (E), that is estimates of the difference between a subsolution and a supersolution in terms of the data of the related problems and the time (see §3).

In §4 we refer to [11] for an existence statement, based on the Perron's method, as regards to the solutions of (1).

Quite possibly more general theorems about existence or/and uniqueness of the solutions of (1) could be established using some refinements from [8], [11] or the results performed by other authors, but this is enough for our aim in this paper.

In §5 we apply the above decay estimates and the above existence results to establish an existence and uniqueness result of time almost periodic (briefly a.p.), or periodic, viscosity solutions of (E) in $\mathbb{R}\times\mathbb{R}^N$ in case $f = f(t)$ is independent of x and a.p. in t, and H is independent of t. We establish also the Lipschitz continuity of the solution. Moreover we state the equivalence between the solvability of the same equation (E) on $\mathbb{R}\times\mathbb{R}^N$ and that of the stationary equation $H(x,u,Du,D^2u) =< f >$ on $\mathbb{R}^N$, where $< f >$ is the average on $\mathbb{R}$ of the a.p. function f.

In this last section we extend to the second order PDEs the results, regarding first order evolution equations, of Bostan and Namah [5], using similar proofs.

It is worthwhile mentioning that part of the results of the present paper are obtained in [20] but for a bounded domain and different proofs.

2. Notations and Preliminaries

Let us start by listing the usual hypothesis used for the existence and/or uniqueness results.

(H_0) $H \in C([0,T)\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N\times S(N))$.

(H_1) (H is proper)

$$H(t,x,r,p,X) \geq H(t,x,s,p,Y)$$

whenever $r \geq s$ and $X \leq Y$, for arbitrary t, x and p. Let us recall that $X \leq Y$ when $< Xh,h > \leq < Yh,h >$ for every $h \in \mathbb{R}^N$.

(H_2)

$$H(t,x,r,p,X) - H(t,x,s,p,X) \geq \gamma(r-s)$$

where γ is a suitable positive constant, for any $r \geq s$, and arbitrary t, x, p, X.

(H_3) There is a continuous and nondecreasing function $\omega : [0,+\infty) \to [0,+\infty)$ such that $\omega(0) = 0$ and

$$H(t,x,r,p,X) - H(t,y,r,q,Y) \leq \omega(|x-y| + |p-q| + \|X-Y\|)$$

for arbitrary t, x, y, r p, q, X, Y.

For a set E, $C(E)$ denotes the set of real valued continuous functions on E and $UC(E)$ denotes the subspace of $C(E)$ consisting of the uniformly continuous functions on E. Let $USC(E)$ (or $LSC(E)$) denote the sets of the real valued upper (or lower) semicontinuous functions on E. Denote $Q_T = (0,T) \times \mathbb{R}^N$ and $R_T = [0,T) \times \mathbb{R}^N$ for $T > 0$.

3. Comparison principle and decay estimate

There are equivalent definitions of viscosity subsolutions or supersolutions. We briefly recall here the ones we will use in the following, referring the reader to [8] for more details on the subject.

Definition 3.1. A function $u \in USC(Q_T)$ is a viscosity subsolution of the equation (E) in Q_T if for all $(t_o,x_o) \in Q_T$ and for all $\phi \in C^2(Q_T)$, if

$$u(t,x) - \phi(t,x) \leq u(t_o,x_o) - \phi(t_o,x_o)$$

for every $(t,x) \in Q_T$, then

$$\phi_t(t_o,x_o) + H(t_o,x_o,u(t_o,x_o),D\phi(t_o,x_o),D^2\phi(t_o,x_o)) \leq f(t_o,x_o)$$

We say that $u \in USC(R_T)$ is a viscosity subsolution of the problem (1) if u is a viscosity subsolution of the equation (E) on Q_T such that $u(0,x) \leq u_o(x)$ for $x \in \mathbb{R}^N$. A function $v \in LSC(R_T)$ is a viscosity supersolution of the equation (E) on Q_T or of the problem (1) if $u = -v$ is a viscosity subsolution of the equation (E) on Q_T or of the problem (1) respectively. A viscosity solution of the equation (E) on Q_T or of the problem (1) is a function $u \in C(R_T)$ which is both viscosity subsolution and supersolution of the equation (E) on Q_T or of the problem (1) respectively.

As in the following we will deal only with viscosity sub(super)solutions or solutions we will omit the term "viscosity" for sake of simplicity.

The following comparison principle can be proved following the outline of [8, Theorem 8.2] but taking into account that, due to the unboundedness of the domain ($\mathbb{R}^N$), we must add to the penalization function a term which goes to infinity when x or y go to infinity, see for examples [2], [4] or [17].

Proposition 3.2 (Comparison principle)**.** *Let H satisfy (H_0), (H_1), (H_3) and let f uniformly continuous on $x \in \mathbb{R}^N$ uniformly w.r.t. $t \in [0,T)$. Let u, v be subsolution and respectively supersolution of* (1) *such that $u - v$ is bounded above. Then $u \leq v$ on R_T.*

Proof. As in [8, Theorem 8.2] we easily recognize that it will simply suffice to prove the comparison under the additional assumptions

$$\begin{aligned} &(i)\ \ u_t + H(t,x,u,Du,D^2u) \leq f(t,x) - \frac{\varepsilon}{T^2}, \\ &(ii)\ \lim_{t\uparrow T} u(t,x) = -\infty \quad \text{uniformly on } \mathbb{R}^N. \end{aligned} \tag{2}$$

Let us suppose by contradiction that there exists $(t_o, x_o) \in [0,T) \times \mathbb{R}^N$ such that

$$u(t_o,x_o) - v(t_o,x_o) = \delta > 0 \tag{3}$$

For any α, $\beta > 0$ let

$$w_{\alpha,\beta}(t,x,y) = u(t,x) - v(t,y) - \frac{\alpha}{2}|x-y|^2 - \frac{\beta}{2}(|x|^2 + |y|^2)$$

and let

$$M_{\alpha,\beta} = \max_{[0,T)\times\mathbb{R}^N\times\mathbb{R}^N} w_{\alpha,\beta}(t,x,y) = w_{\alpha,\beta}(t_{\alpha,\beta}, x_{\alpha,\beta}, y_{\alpha,\beta}).$$

Such a maximum exists because $w_{\alpha,\beta}(t,x,y) \to -\infty$ if $|x| + |y| \to +\infty$ or $t \uparrow T$. Moreover $u - v$ is bounded above and upper semicontinuous. By (3), $M_{\alpha,\beta} \geq \delta - \beta|x_o|^2$. Let β be so small that $\delta - \beta|x_o|^2 \geq \sigma > 0$

Using [2, Lemma 2.9] we have for $\alpha \to +\infty$ and $\beta \to 0$

$$\frac{\alpha}{2}|x_{\alpha,\beta} - y_{\alpha,\beta}|^2 + \frac{\beta}{2}(|x_{\alpha,\beta}|^2 + |y_{\alpha,\beta}|^2) \to 0. \tag{4}$$

If $t_{\alpha,\beta} = 0$ for $\alpha \to +\infty$ and $\beta \to 0$ (or this holds for a subsequence of $t_{\alpha,\beta}$) then we have

$$\begin{aligned} 0 < \sigma &\le M_{\alpha,\beta} \\ &\le \sup_{\mathbb{R}^N \times \mathbb{R}^N} u_o(x_{\alpha,\beta}) - u_o(y_{\alpha,\beta}) - \frac{\alpha}{2}|x_{\alpha,\beta} - y_{\alpha,\beta}|^2 - \frac{\beta}{2}(|x_{\alpha,\beta}|^2 + |y_{\alpha,\beta}|^2) \end{aligned} \tag{5}$$

and we obtain a contradiction for $\alpha \to +\infty$ and $\beta \to 0$ in virtue of (4).

Then $t_{\alpha,\beta} > 0$ for $\alpha \to +\infty$ and $\beta \to 0$. Applying [8, Theorem 8.3] at $(t_{\alpha,\beta}, x_{\alpha,\beta}, y_{\alpha,\beta})$ we obtain the relations (if $C = \frac{\varepsilon}{T^2}$)

$$\begin{aligned} &a + H(t_{\alpha,\beta}, x_{\alpha,\beta}, u(t_{\alpha,\beta}, x_{\alpha,\beta}), \alpha(x_{\alpha,\beta} - y_{\alpha,\beta}) + \beta x_{\alpha,\beta}, X) \le -C + f(t_{\alpha,\beta}, x_{\alpha,\beta}) \\ &a + H(t_{\alpha,\beta}, y_{\alpha,\beta}, v(t_{\alpha,\beta}, y_{\alpha,\beta}), \alpha(x_{\alpha,\beta} - y_{\alpha,\beta}) - \beta y_{\alpha,\beta}, Y) \ge f(t_{\alpha,\beta}, y_{\alpha,\beta}) \end{aligned} \tag{6}$$

for suitable $a \in \mathbb{R}$ and X, $Y \in S(N)$ satisfying the condition $X \le Y + O(\beta)$ for $\beta \to 0$. Let us observe that

$$u(t_{\alpha,\beta}, x_{\alpha,\beta}) \ge v(t_{\alpha,\beta}, y_{\alpha,\beta}) \tag{7}$$

because otherwise $M_{\alpha,\beta} < 0$. From (6), (7) and taking into account the assumptions (H_1) and (H_3) we obtain

$$\begin{aligned} &C + f(t_{\alpha,\beta}, y_{\alpha,\beta}) - f(t_{\alpha,\beta}, x_{\alpha,\beta}) \\ &\quad \le H(t_{\alpha,\beta}, y_{\alpha,\beta}, v(t_{\alpha,\beta}, y_{\alpha,\beta}), \alpha(x_{\alpha,\beta} - y_{\alpha,\beta}) - \beta y_{\alpha,\beta}, Y) \\ &\quad - H(t_{\alpha,\beta}, x_{\alpha,\beta}, u(t_{\alpha,\beta}, x_{\alpha,\beta}), \alpha(x_{\alpha,\beta} - y_{\alpha,\beta}) + \beta x_{\alpha,\beta}, X) \\ &\quad \le H(t_{\alpha,\beta}, y_{\alpha,\beta}, u(t_{\alpha,\beta}, x_{\alpha,\beta}), \alpha(x_{\alpha,\beta} - y_{\alpha,\beta}) - \beta y_{\alpha,\beta}, X + O(\beta)) \\ &\quad - H(t_{\alpha,\beta}, x_{\alpha,\beta}, u(t_{\alpha,\beta}, x_{\alpha,\beta}), \alpha(x_{\alpha,\beta} - y_{\alpha,\beta}) + \beta x_{\alpha,\beta}, X) \\ &\quad \le \omega(|x_{\alpha,\beta} - y_{\alpha,\beta}| + \beta|x_{\alpha,\beta} + y_{\alpha,\beta}| + \|O(\beta)\|) \to 0 \end{aligned} \tag{8}$$

if $\alpha \to +\infty$ and $\beta \to 0$. As $f(t_{\alpha,\beta}, y_{\alpha,\beta}) - f(t_{\alpha,\beta}, x_{\alpha,\beta}) \to 0$ when $\alpha \to +\infty$ and $\beta \to 0$ we have again a contradiction. This completes the proof. □

Theorem 3.3 (Decay estimate)**.** *Let H satisfy (H_0), ..., (H_3). Let $f^1, f^2 \in C([0,T) \times \mathbb{R}^N)$ be bounded and uniformly continuous in $x \in \mathbb{R}^N$ uniformly w.r.t. $t \in [0,T)$. Let u be a bounded subsolution of*

$$u_t + H(t,x,u,Du,D^2u) = f^1(t,x), \ (t,x) \in Q_T$$

and let v be a bounded supersolution of

$$v_t + H(t,x,v,Dv,D^2v) = f^2(t,x), \ (t,x) \in Q_T.$$

Then for every $t \in [0,T)$ we have

$$e^{\gamma t} \sup_{x\in\mathbb{R}^N} (u(t,x)-v(t,x))_+ \le \|\,(u(0,\cdot)-v(0,\cdot))_+\,\|_{L^\infty(\mathbb{R}^N)} \tag{9}$$

$$+\int_0^t e^{\gamma s}\|\left(f^1(s,\cdot)-f^2(s,\cdot)\right)_+\|_{L^\infty(\mathbb{R}^N)}\,ds.$$

Proof. For $(t,x)\in Q_T$, set $w^1(t,x):=e^{\gamma t}u(t,x)$ and $w^2(t,x):=e^{\gamma t}v(t,x)+A(t)$, where

$$A(t):=\|\,(u(0,\cdot)-v(0,\cdot))_+\,\|_{L^\infty(\mathbb{R}^N)}+\int_0^t e^{\gamma s}\|\left(f^1(s,\cdot)-f^2(s,\cdot)\right)_+\|_{L^\infty(\mathbb{R}^N)}\,ds$$

It is not hard to see that w^1 and w^2 are a subsolution and resp. a supersolution of the equation

$$\varphi_t(t,x)+\widetilde{H}(t,x,\varphi,D\varphi,D^2\varphi)=e^{\gamma t}f^1(t,x)$$

in $(0,T)\times\mathbb{R}^N$, where $\widetilde{H}(t,x,\varphi,D\varphi,D^2\varphi)=e^{\gamma t}H(t,x,e^{-\gamma t}\varphi,e^{-\gamma t}D\varphi,e^{-\gamma t}D^2\varphi)$ $-\gamma\varphi$ and where $\widetilde{H}(t,x,\varphi,p,X)=e^{\gamma t}H(t,x,e^{-\gamma t}\varphi,e^{-\gamma t}p,e^{-\gamma t}X)-\gamma\varphi$ satisfies (H_0), (H_1) and (H_3).

In fact, about w^1, let (t_o,x_o) be a maximum point for $w^1-\psi$, where $\psi\in C^2(Q_T)$. We can suppose $\psi(t,x)=e^{\gamma t}\alpha(t,x)$ where $\alpha\in C^2(Q_T)$. So $w^1-\psi=e^{\gamma t}(u-\alpha)$. Let $\psi_o(t,x)=\psi(t,x)+L$ where $L=w^1(t_o,x_o)-\psi(t_o,x_o)$. So (t_o,x_o) is a maximum point for $w^1-\psi_o$ and $(w^1-\psi_o)(t,x)\le(w^1-\psi_o)(t_o,x_o)=0$. Let $\alpha_o=\alpha+Le^{-\gamma t}$. Then $w^1-\psi_o=(u-\alpha_o)e^{\gamma t}$ and $(u-\alpha_o)(t,x)=e^{-\gamma t}(w^1-\psi_o)(t,x)\le 0\le e^{-\gamma t_o}(w^1-\psi_o)(t_o,x_o)=(u-\alpha_o)(t_o,x_o)$, that is (t_o,x_o) is a maximum point also for $(u-\alpha_o)$. The definition of subsolution implies that

$$(\alpha_o)_t(t_o,x_o)+H(t_o,x_o,u(t_o,x_o),D\alpha_o(t_o,x_o),D^2\alpha_o(t_o,x_o))\le f^1(t_o,x_o)$$

that is, as $\alpha_o=e^{-\gamma t}\psi_o$,

$$\psi_t(t_o,x_o)+e^{\gamma t_o}H(t_o,x_o,e^{-\gamma t_o}w^1(t_o,x_o),e^{-\gamma t_o}D\psi(t_o,x_o),e^{-\gamma t_o}D^2\psi(t_o,x_o))$$

$$-\gamma w^1(t_o,x_o)\le e^{\gamma t_o}f^1(t_o,x_o).$$

With regard to w^2 take into account that, in virtue of (H_2), we have

$$e^{\gamma t_o}H(t_o,x_o,e^{-\gamma t_o}(w^2(t_o,x_o)-A(t_o)),e^{-\gamma t_o}D\psi(t_o,x_o),e^{-\gamma t_o}D^2\psi(t_o,x_o))$$

$$\le e^{\gamma t_o}H(t_o,x_o,e^{-\gamma t_o}w^2(t_o,x_o),e^{-\gamma t_o}D\psi(t_o,x_o),e^{-\gamma t_o}D^2\psi(t_o,x_o))-\gamma A(t_o)$$

where (t_o,x_o) is a minimum point for $(w^2-A)-\psi$, i.e. for $w^2-\varphi$, where $\varphi=\psi+A$.

Let us observe that $D\psi = D\varphi$, $D^2\psi = D^2\varphi$, $\psi_t = \varphi_t - A_t$ and, as $A_t \geq e^{\gamma t} \| \left(f^1(t,\cdot) - f^2(t,\cdot)\right)_+ \|_{L^\infty(\mathbb{R}^N)}$, then

$$e^{\gamma t_o} f^2(t_o,x_o) + A_t(t_o) \geq e^{\gamma t_o} f^1(t_o,x_o).$$

It is also clear that $w^1(0,x) \leq w^2(0,x)$ on $\mathbb{R}^N$. By the comparison principle, Proposition 3.2, we get

$$w^1(t,x) \leq w^2(t,x) \text{ on } R_T,$$

and the conclusion follows. □

Corollary 3.4. *Let the hypothesis of Theorem 3.3 be in force. Then for all* $t \in [0,T)$

$$\sup_{x\in\mathbb{R}^N} (u(t,x) - v(t,x)) \leq e^{-\gamma t} \| (u(0,\cdot) - v(0,\cdot))_+ \|_{L^\infty(\mathbb{R}^N)}$$
$$+ \sup_{0\leq s\leq t} \int_s^t \sup_{x\in\mathbb{R}^N} \left(f^1(\sigma,x) - f^2(\sigma,x)\right) d\sigma. \quad (10)$$

Proof. Let us fix $t \in [0,T)$. We denote by $h : [0,T) \to \mathbb{R}$ the function $h(\sigma) := \sup_{x\in\mathbb{R}^N} \left(f^1(\sigma,x) - f^2(\sigma,x)\right)$. Consider the function $w : [0,T) \times \mathbb{R}^N \to \mathbb{R}$ given by

$$w(s,x) := v(s,x) + \int_0^s h(\sigma)\, d\sigma + \sup_{0\leq\tau\leq t} \left(-\int_0^\tau h(\sigma)\, d\sigma\right).$$

Let us denote for $s \in [0,T)$

$$A(s) := \int_0^s h(\sigma)\, d\sigma + \sup_{0\leq\tau\leq t} \left(-\int_0^\tau h(\sigma)\, d\sigma\right).$$

It is easily seen that w is bounded because v and A are bounded. Moreover as in the proof of Theorem 3.3 we can prove that w is a supersolution of $\partial_s + H = f^1$, $(s,x) \in (0,t) \times \mathbb{R}^N$. In fact if $(s_o,x_o) \in (0,t) \times \mathbb{R}^N$ is a minimum point for $w - \varphi$, where φ is a regular function in $(0,t) \times \mathbb{R}^N$, i.e. for $(w-A) - (\varphi - A) = v - \psi$, where $\psi = \varphi - A$, then (as v is a supersolution) we give

$$\psi_s(s_o,x_o) + H(s_o,x_o,v(s_o,x_o),D\psi(s_o,x_o),D^2\psi(s_o,x_o)) \geq f^2(s_o,x_o)$$

As $v = w - A$, in virtue of (H_2) and being $\psi_s = \varphi_s - A_s$, $D\psi = D\varphi$, $D^2\psi = D^2\varphi$ we obtain

$$\varphi_s(s_o,x_o) + H(s_o,x_o,w(s_o,x_o),D\varphi(s_o,x_o),D^2\varphi(s_o,x_o))$$
$$\geq \gamma A(s_o) + A_s(s_o) + f^2(s_o,x_o) \geq f^1(s_o,x_o).$$

We deduce from Theorem 3.3 that for any $(s,x) \in (0,t) \times \mathbb{R}^N$

$$e^{\gamma s}(u(s,x) - w(s,x)) \leq \sup_{x \in \mathbb{R}^N} (u(0,x) - w(0,x))_+ \leq \sup_{x \in \mathbb{R}^N} (u(0,x) - v(0,x))_+$$

implying that

$$u(s,x) - v(s,x) \leq e^{-\gamma s} \sup_{x \in \mathbb{R}^N} (u(0,x) - v(0,x))_+ + \int_0^s h(\sigma)\, d\sigma \\ + \sup_{0 \leq \tau \leq t} \left(- \int_0^\tau h(\sigma)\, d\sigma \right).$$

In particular (eventually changing t with $t + \varepsilon$ for a small ε) for $s = t$ one gets for any $x \in \mathbb{R}^N$

$$u(t,x) - v(t,x) \leq e^{-\gamma t} \sup_{x \in \mathbb{R}^N} (u(0,x) - v(0,x))_+ \\ + \sup_{0 \leq \tau \leq t} \left(\int_\tau^t \sup_{x \in \mathbb{R}^N} \left(f^1(\sigma,x) - f^2(\sigma,x) \right) d\sigma \right).$$

□

Note that in the right hand side term of (10) we have now $\sup_{x \in \mathbb{R}^N} \big(f^1(\sigma,x) - f^2(\sigma,x) \big)$ and not $\sup_{x \in \mathbb{R}^N} \big(f^1(\sigma,x) - f^2(\sigma,x) \big)_+$.

Corollary 3.5. *Let H satisfy (H_0), ..., (H_3). Let f^1, $f^2 \in C(\mathbb{R} \times \mathbb{R}^N)$ bounded and uniformly continuous on x uniformly w.r.t. t. Let u be a bounded subsolution of*

$$u_t + H(t,x,u,Du,D^2u) = f^1(t,x)$$

in $\mathbb{R} \times \mathbb{R}^N$, and let v be a bounded supersolution of

$$v_t + H(t,x,v,Du,D^2v) = f^2(t,x)$$

in $\mathbb{R} \times \mathbb{R}^N$. Then for every $t \in \mathbb{R}$ we have

$$\sup_{x \in \mathbb{R}^N} (u(t,x) - v(t,x)) \leq e^{-\gamma t} \int_{-\infty}^t e^{\gamma s} \| \left(f^1(s,\cdot) - f^2(s,\cdot) \right)_+ \|_{L^\infty(\mathbb{R}^N)}\, ds. \quad (11)$$

Proof. Take t_o, $t \in \mathbb{R}$, $t_o \leq t$ and using the proof of Theorem 3.3 write for all $x \in \mathbb{R}^N$

$$u(t,x) - v(t,x) \leq e^{-\gamma(t - t_o)} \cdot (\|u\|_\infty + \|v\|_\infty) \\ + e^{-\gamma t} \int_{t_o}^t e^{\gamma s} \| \left(f^1(s,\cdot) - f^2(s,\cdot) \right)_+ \|_{L^\infty(\mathbb{R}^N)}\, ds.$$

The conclusion follows by letting $t_o \to -\infty$. □

Corollary 3.6. *Let the hypothesis of Corollary 3.5 be in force. Then for every* $t \in \mathbb{R}$

$$\sup_{x\in\mathbb{R}^N} (u(t,x) - v(t,x)) \le \sup_{s\le t} \int_s^t \sup_{x\in\mathbb{R}^N} \left(f^1(\sigma,x) - f^2(\sigma,x)\right) d\sigma. \tag{12}$$

Proof. Take $t_o, t \in \mathbb{R}$, $t_o \le t$. Repeating the proof of Corollary 3.4 in $[t_o,t]\times\mathbb{R}^N$ we obtain

$$u(t,x) - v(t,x) \le e^{-\gamma(t-t_o)} \sup_{x\in\mathbb{R}^N} (u(t_o,x) - v(t_o,x))_+ \\ + \sup_{t_o\le\tau\le t} \left(\int_\tau^t \sup_{x\in\mathbb{R}^N} \left(f^1(\sigma,x) - f^2(\sigma,x)\right) d\sigma \right).$$

The conclusion follows by letting $t_o \to -\infty$. □

4. Existence

We say that a continuous nondecreasing function $m : [0,+\infty) \to [0,+\infty)$ is a modulus if $m(0) = 0$ and $m(r+s) \le m(r) + m(s)$ for any r, $s \ge 0$. Let $T > 0$ and let $UC_s(R_T)$ denote the space of those $u \in C(R_T)$ for which there is a modulus m and $r > 0$ such that

$$|u(t,x) - u(t,y)| \le m(|x-y|)$$

for (t,x), $(t,y) \in R_T$ with $|x-y| \le r$.

The following propositions establishes conditions under which the equation (E) or the problem (1) admits a solution. We refer here to the results of [11] for any proof pointing out that the assumptions $(H_0),\ldots,(H_3)$ and the request that $f \in C(R_T)$ is Lipschitz continuous in x uniformly w.r.t. t, imply the assumptions $(F_0),\ldots,(F_4)$ of [11]. We limit ourselves to remark that the proofs of [11] are based on the Perron's method (see to this end also the papers [12], [13], [14], [8]).

Proposition 4.1 ([11], Theorem 1, (ii))**.** *Let H satisfy $(H_0),\ldots,(H_3)$ where ω in (H_3) is a modulus, and let $f \in C(R_T)$ be Lipschitz continuous in x uniformly w.r.t. t. Let $\underline{u}$ and $\bar{u}$ be respectively a subsolution and a supersolution of the equation (E) such that*

$$\underline{u},\, \bar{u} \in UC_s(R_T),\ \ \underline{u} \le \bar{u} \text{ in } Q_T.$$

Then there exists a solution u of the equation (E) such that $\underline{u} \le u \le \bar{u}$ in Q_T.

If we suppose also $\underline{u}(0,x) = \bar{u}(0,x)$ for every $x \in \mathbb{R}^N$, then there exists a unique viscosity solution $u \in UC_s(R_T)$ of the equation (E) which satisfies $\underline{u} \le u \le \bar{u}$ in Q_T and $u(0,x) = \underline{u}(0,x)$ for every $x \in \mathbb{R}^N$.

Proposition 4.2 ([11], Corollary 2)**.** *Let H and f satisfy the assumptions of the above Proposition 4.1, and let $u_o \in UC(\mathbb{R}^N)$. Then the Cauchy problem* (1) *has a unique solution $u \in UC_s(R_T)$.*

Remark 4.3. In virtue of the Proposition 3.2 we can also affirm that, if $\underline{u}$ and $\bar{u}$ are respectively a subsolution and a supersolution of the equation (E) such that $\underline{u}$, $\bar{u} \in UC_s(R_T)$ and $\underline{u} \leq u_0 \leq \bar{u}$ on $\{0\} \times \mathbb{R}^N$ where $u_o \in UC(\mathbb{R}^N)$, then the unique solution u of the problem (1) stated in Proposition 4.2 satisfies $\underline{u} \leq u \leq \bar{u}$ on R_T.

Remark 4.4. Let H be defined for $t \in [0,+\infty)$ with the properties required in Proposition 4.2. Taking into account the arbitrariness of $T > 0$, in virtue of Proposition 4.2 we can affirm the existence and uniqueness of a solution $u \in UC_s([0,T) \times \mathbb{R}^N)$ for any $T > 0$, of the problem

$$\begin{cases} u_t + H(t,x,u,Du,D^2u) = f(t,x), & (t,x) \in (0,+\infty) \times \mathbb{R}^N \\ u(0,x) = u_o(x), & x \in \mathbb{R}^N. \end{cases} \tag{13}$$

under the conditions $u_o \in UC(\mathbb{R}^N)$ and $f \in C((0,+\infty) \times \mathbb{R}^N)$ Lipschitz continuous in x uniformly w.r.t. t.

5. Application : existence of almost periodic viscosity solutions

We are now in a position to state the existence and uniqueness of almost periodic (periodic) viscosity solutions of the equation

$$u_t + H(x,u,Du,D^2u) = f(t) \quad , (t,x) \in \mathbb{R} \times \mathbb{R}^N \tag{14}$$

where H is independent of t and f is continuous and almost periodic (briefly a.p.) or periodic.

5.1. Almost periodic functions

In this subsection we recall the definition and some fundamental properties of almost periodic functions. For more details one can refer to [6], [10].

Definition 5.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. We say that f is almost periodic if it satisfies the following condition

$$\forall \varepsilon > 0 \, \exists l(\varepsilon) > 0 \textit{ such that } \forall a \in \mathbb{R} \, \exists \tau \in [a, a + l(\varepsilon)) \textit{ satisfying}$$

$$|f(t+\tau) - f(t)| < \varepsilon \, , \, \forall t \in \mathbb{R} \tag{15}$$

A number τ verifying (15) is called ε almost period.

Proposition 5.2. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is almost periodic. Then*

i) *f is bounded and uniformly continuous in $\mathbb{R}$.*

ii) $(1/T)\int_a^{a+T} f(t)\,dt$ *converges as $T \to +\infty$ uniformly with respect to $a \in \mathbb{R}$. The limit is called the average of f and denoted*

$$< f >:= \lim_{T\to+\infty} \frac{1}{T}\int_a^{a+T} f(t)\,dt\ ,\ uniformly\ w.r.t.\ a \in \mathbb{R}$$

If f is periodic then $< f >$ denotes the usual definition of mean of f over one period.

iii) *If F denotes a primitive of f, then F is almost periodic if and only if F is bounded.*

The following definition extends the notion of almost periodicity in order to apply it to differential equations [19].

Definition 5.3. We say that $u : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is almost periodic in t uniformly with respect to x if u is continuous in t uniformly with respect to x and

$$\forall \varepsilon > 0\, \exists\, l(\varepsilon) > 0 \textit{ such that } \forall a \in \mathbb{R}\, \exists \tau \in [a, a + l(\varepsilon)) \ \textit{ satisfying}$$

$$|u(t+\tau, x) - u(t,x)| < \varepsilon\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

5.2. Existence of a.p. (periodic) solutions

For any interval $I \subseteq \mathbb{R}$ we will denote by $BUC(I \times \mathbb{R}^N)$ the set of all real functions which are bounded and uniformly continuous on $I \times \mathbb{R}^N$, equipped with the uniform norm.

Theorem 5.4. *Assume that $H : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times S(N) \to \mathbb{R}$ (H is independent of t) satisfies $(H_0), \ldots, (H_3)$ where ω in (H_3) is a modulus. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous a.p. (periodic) function and assume that there exists a constant $M > 0$ such that*

$$H(x, -M, 0, 0) \leq f(t) \leq H(x, M, 0, 0) \tag{16}$$

for every $(t,x) \in \mathbb{R} \times \mathbb{R}^N$.

Then there is a unique solution $u \in BUC(\mathbb{R} \times \mathbb{R}^N)$ of (14) *which is a.p. (periodic) in $t \in \mathbb{R}$ uniformly w.r.t. $x \in \mathbb{R}^N$.*

Proof. First we prove the existence of the solution. For any integer $n \geq 0$ let u_n be the solution $u_n \in UC_s([-n,T) \times \mathbb{R}^N)$ for any $T > -n$ of the problem

$$\begin{cases} u_t + H(x,u,Du,D^2u) = f(t)\,, & (t,x) \in (-n,+\infty) \times \mathbb{R}^N \\ u(-n,x) = 0\,, & x \in \mathbb{R}^N \end{cases} \tag{17}$$

Let us observe that by virtue of hypothesis (16), on account of (17), Remark 4.3 and Remark 4.4, we have

$$-M \leq u_n(t,x) \leq M,\ (t,x) \in [-n,+\infty) \times \mathbb{R}^N.$$

Take $t \in \mathbb{R}$ and, for $m \geq n$ large enough, by Corollary 3.4 we can write for all $x \in \mathbb{R}^N$ and $t \geq t_o \geq -n$

$$|u_n(t,x) - u_m(t,x)| \leq e^{-\gamma(t-t_o)}\,(\|u_n\|_\infty + \|u_m\|_\infty) \leq 2Me^{-\gamma(t-t_o)}$$

For $t_o = -n$ we deduce $|u_n(t,x) - u_m(t,x)| \leq 2Me^{-\gamma t}e^{-\gamma n}$ and thus there exists

$$\lim_{n\to+\infty} u_n(t,x) = u(t,x)\ ,\ \text{for every}\ (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

Moreover $(u_n)_n$ converges uniformly on $[a,+\infty) \times \mathbb{R}^N$ for every $a \in \mathbb{R}$.

By using the stability result for continuous viscosity solutions [2], [8], we deduce that u verifies (14) in the viscosity sense.

As $u_n \in UC_s([-n,T) \times \mathbb{R}^N)$, for any $T > -n$, then $u_n(t.\cdot) \in UC(\mathbb{R}^N)$ uniformly w.r.t. $t \in [-n,T)$. Then for every a, $b \in \mathbb{R}$, with $a < b$, for n large enough, $u_n(t,\cdot) \in UC(\mathbb{R}^N)$ uniformly w.r.t. $t \in [a,b]$. So for every a, $b \in \mathbb{R}$, with $a < b$, $u(t,\cdot) \in UC(\mathbb{R}^N)$ uniformly w.r.t. $t \in [a,b]$. Moreover we will prove that $u(\cdot,x)$ is a.p. (periodic) in $t \in \mathbb{R}$ (then uniformly continuous in t) uniformly w.r.t. $x \in \mathbb{R}^N$. So $u \in BUC([a,b] \times \mathbb{R}^N)$ for every a, $b \in \mathbb{R}$, with $a < b$, and by almost periodicity, $u \in BUC(\mathbb{R} \times \mathbb{R}^N)$.

To prove the almost periodicity fix an arbitrary $\varepsilon > 0$ and consider $l(\gamma\varepsilon)$ such that any interval of length $l(\varepsilon\gamma)$ contains a $\gamma\varepsilon$ almost period of f. We will show that any interval of length $l(\varepsilon\gamma)$ contains a number τ which is an ε-almost period for $u(\cdot,x)$, for every $x \in \mathbb{R}^N$.

Indeed, consider an interval of length $l(\varepsilon\gamma)$, take τ a $\gamma\varepsilon$-almost period of f and let us fix $\tilde{t} \in \mathbb{R}$. Observe that the function $v_n : [-n-\tau,+\infty) \times \mathbb{R}^N \to \mathbb{R}$, $v_n(t,x) := u_n(t+\tau,x)$ solves in the viscosity sense

$$\partial_t v_n + H(x,v_n,Dv_n,D^2v_n) = f(t+\tau)\ ,\ (t,x) \in (-n-\tau,+\infty) \times \mathbb{R}^N$$

By Theorem 3.3 we have for all $t \geq t_n = \max\{-n,-n-\tau\}$

$$\begin{aligned} |u_n(t,x) - v_n(t,x)| \leq{}& e^{-\gamma(t-t_n)}\,(\|u_n\|_\infty + \|v_n\|_\infty) \\ &+ e^{-\gamma(t-t_n)} \int_{t_n}^{t} e^{\gamma\sigma} |f(\sigma+\tau) - f(\sigma)|\, d\sigma \end{aligned} \tag{18}$$

In particular, for $t = \tilde{t}$ and n large enough, we obtain

$$|u_n(\tilde{t},x) - u_n(\tilde{t}+\tau,x)| \leq 2Me^{-\gamma(\tilde{t}-t_n)} + e^{-\gamma\tilde{t}} \int_{t_n}^{\tilde{t}} e^{\gamma\sigma}\gamma\varepsilon\, d\sigma \leq 2Me^{-\gamma(\tilde{t}-t_n)} + \varepsilon.$$

By letting $n \to +\infty$ we have $t_n \to -\infty$ and therefore

$$|u(\tilde{t},x) - u(\tilde{t}+\tau,x)| \leq \varepsilon \ , \ (\tilde{t},x) \in \mathbb{R} \times \mathbb{R}^N.$$

In case of f periodic of period $T > 0$ the same calculations give

$$|u_n(t,x) - u_n(t+T,x)| \leq 2Me^{-\gamma(t+nT)}$$

for any $x \in \mathbb{R}^N$ and $t \geq t_n = -nT$, in place of (18), and the result follows again by letting $n \to +\infty$. □

Theorem 5.5. *Let the assumptions of Theorem 5.4 be in force. Assume also that*

(H_4) $\lim_{|p|\to+\infty} H(x,r,p,X) = +\infty$

uniformly w.r.t. x, r, X. Then the solution u of (14) *is also Lipschitz continuous on* $\mathbb{R} \times \mathbb{R}^N$.

Proof. Let u_n be the solution of (17) and let $h \in \mathbb{R}$. Observe that, from the autonomous character of the Hamiltonian H, $v_n(t,x) := u_n(t+h,x)$ solves in the viscosity sense

$$\partial_t v_n + H(x, v_n, Dv_n, D^2 v_n) = f(t+h) \ , \ (t,x) \in [-n-h, +\infty) \times \mathbb{R}^N$$

By Corollary 3.4 we have, for all $t \geq t_n = \max\{-n, -n-h\}$, $x \in \mathbb{R}^N$

$$|u_n(t,x) - u_n(t+h,x)| \leq 2Me^{-\gamma(t-t_n)} + \sup_{s\in[t_n,t]} \int_s^t [f(\sigma+h) - f(\sigma)]\, d\sigma$$

where

$$\sup_{s\in[t_n,t]} \int_s^t [f(\sigma+h) - f(\sigma)]\, d\sigma = \sup_{s\in[t_n,t]} \{\int_t^{(t+h)} f(\sigma)\, d\sigma - \int_s^{(s+h)} f(\sigma)\, d\sigma\}$$
$$\leq 2|h|\|f\|_\infty$$

and then

$$|u_n(t,x) - u_n(t+h,x)| \leq 2Me^{-\gamma(t-t_n)} + 2|h|\|f\|_\infty$$

If $n \to +\infty$, then we obtain $|u_n(t,x) - u_n(t+h,x)| \leq 2|h|\|f\|_\infty$ for every $(t,x) \in \mathbb{R} \times \mathbb{R}^N$ and $h \in \mathbb{R}$. So u is Lipschitz continuous w.r.t. $t \in \mathbb{R}$ uniformly w.r.t. $x \in \mathbb{R}^N$, and then u_t is bounded in the viscosity sense. But u solves (14). So, by assumption (H_4) we obtain that Du is bounded in the viscosity sense and then u is also Lipschitz w.r.t. $x \in \mathbb{R}^N$ uniformly w.r.t. $t \in \mathbb{R}$ (ref [4], Proposition 3.6). It follows that u is Lipschitz continuous in $\mathbb{R} \times \mathbb{R}^N$. □

The following theorem establishes a close relation between the existence of time almost periodic (periodic) viscosity solutions of (14) and that of stationary viscosity solutions for the time averaged equation

$$H(x,u,Du,D^2u) = < f > \text{ in } \mathbb{R}^N \tag{19}$$

Theorem 5.6. *Assume that H satisfies (H_0), (H_1), (H_3), (H_4) (the assumption (H_2) is unnecessary in this case) and* $\sup\{|H(x,0,0,0)| : x \in \mathbb{R}^N\} = C < +\infty$. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is an almost periodic (periodic) function such that $F(t) := \int_0^t \{f(\sigma) - < f >\} d\sigma$ is bounded on $\mathbb{R}$. Then there is a bounded Lipschitz time almost periodic (periodic) viscosity solution of (14) if and only if there is a bounded Lipschitz viscosity solution of (19).*

Proof. The proof is substantially the same of [5, Theorem 8], and uses Proposition 5.2 and Corollary 3.6. □

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SILVANA MARCHI
Dipartimento di Matematica e Informatica
Università degli Studi di Parma
V.le Parco Area delle Scienze 53/A
43124 Parma, Italy
e-mail: silvana.marchi@unipr.it